Sparsity-Promoting Dynamic Mode Decomposition with Control

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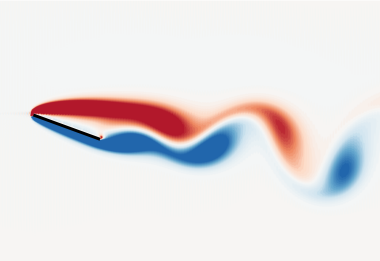
High-Dimensional Systems

- Systems with thousands/millions of states
- Example: Fluid Flows
  - Infinite-dimensional systems
  - Governed by the Navier-Stokes partial differential equations:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\, \boldsymbol{u}) = 0$$

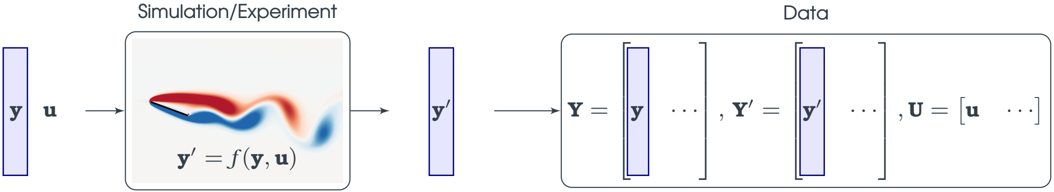
$$\frac{\partial}{\partial t}(\rho\, \boldsymbol{u}) + \nabla \cdot (\rho\, \boldsymbol{u} \otimes \boldsymbol{u}) = -\nabla p + \nabla \cdot \boldsymbol{\tau} + \rho\, \boldsymbol{g}$$



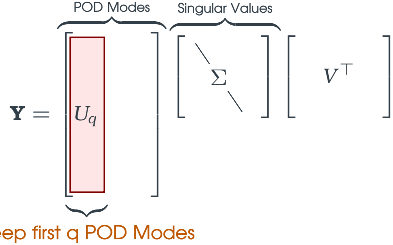
- Dynamic Mode Decomposition:
  - Find a low-dimensional state space
  - Approximate low-dimensional dynamics with a linear system (easier to control)
  - Use only data

Dynamic Mode Decomposition: Model Reduction

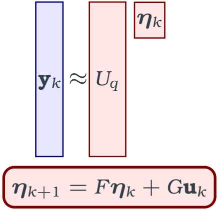
- Collect data:



- Proper Orthogonal Decomposition



- Low-dimensional linear subspace spanned by the columns of U_q:



Dynamic Mode Decomposition: Linear Dynamics

- Fit linear dynamics for $\boldsymbol{\eta}_k$:

$$\begin{aligned}
\boldsymbol{\eta}_{k+1} &= F\boldsymbol{\eta}_k + G\mathbf{u}_k \Rightarrow \\
U_q^\top \mathbf{y}_{k+1} &= FU_q^\top \mathbf{y}_k + G\mathbf{u}_k \Rightarrow \\
U_q^\top \mathbf{Y}' &= FU_q^\top \mathbf{Y} + G\mathbf{U} \Rightarrow \\
\begin{bmatrix} F & G \end{bmatrix} &= U_q^\top \mathbf{Y}' \begin{bmatrix} U_q^\top \mathbf{Y} \\ \mathbf{U} \end{bmatrix}^\dagger
\end{aligned}$$

- Eigenvalue decomposition of F:

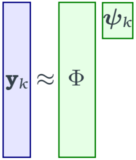
$$FW = W\Lambda$$

  - Eigenvectors: $W = \begin{bmatrix} \mathbf{w}_1 & \cdots & \mathbf{w}_q \end{bmatrix}$
  - Eigenvalues: $\Lambda = \mathrm{diag}\{\begin{bmatrix} \lambda_1 \cdots \lambda_q \end{bmatrix}\}$

- DMD Modes:

$$\Phi = U_q W$$

- DMD Dynamics:



$$\psi_{k+1} = \Lambda\psi_k + \Gamma\mathbf{u}_k$$

where

$$\psi_k = W^{-1}\boldsymbol{\eta}_k \approx W^{-1}U_q y_k$$

$$\Gamma = W^{-1}G$$

POD vs DMD

- POD Modes:

$$U_q = \begin{bmatrix} | & & | \\ \mathbf{v}_1 & \cdots & \mathbf{v}_q \\ | & & | \end{bmatrix} \in \mathbb{R}^{n_y \times q}$$

$$\begin{aligned} \eta_{k+1} &= F\eta_k + G\mathbf{u}_k \\ \mathbf{y}_k &\approx U_q \eta_k \end{aligned}$$

  - Real vectors
  - Modes decompose space based on energy ($\mathscr{L}_2$ norm)
  - Ordered by their energy content

- DMD Modes:

$$\Phi = \begin{bmatrix} | & & | \\ \phi_1 & \cdots & \phi_q \\ | & & | \end{bmatrix} \in \mathbb{C}^{n_y \times q}$$

$$\begin{aligned} \psi_{k+1} &= \Lambda\psi_k + \Gamma\mathbf{u}_k \\ \mathbf{y}_k &\approx \Phi\psi_k \end{aligned}$$

  - Complex (in general) vectors
  - Modes decompose space based on frequency and growth/decay rate (eigenvalues)
  - No apparent ordering

Sparsity-Promoting DMD with Control*

- Since Λ is diagonal, we can write:

$$\mathbf{y}_{k+1} \approx \Phi(\Lambda\psi_k + \Gamma\mathbf{u}_k) = \sum_{i=1}^{q} \phi_i \left(\lambda_i \psi_{i,k} + \Gamma_{i,:}\mathbf{u}_k\right)$$

- Weight the contribution of each DMD mode ϕ_i by $\alpha_i = 1$:

$$\mathbf{y}_{k+1} \approx \sum_{i=1}^{q} \alpha_i \phi_i \left(\lambda_i \psi_{i,k} + \Gamma_{i,:}\mathbf{u}_k\right)$$

- Stack everything together:

$$\mathbf{Y}' \approx \Phi\mathrm{diag}\{\boldsymbol{\alpha}\}\mathbf{R}, \quad \mathbf{R} = \Lambda\Phi^{\dagger}\mathbf{Y} + \Gamma\mathbf{U}$$

- Sparsity-Promoting Optimization:

$$\min_{\boldsymbol{\alpha}} \left\|\mathbf{Y}' - \Phi\mathrm{diag}\{\boldsymbol{\alpha}\}\mathbf{R}\right\|_F^2 + \varepsilon \left\|\boldsymbol{\alpha}\right\|_0$$

  - Approximate data with linear dynamics
  - Promote sparsity: approximate $\mathscr{L}_0$ norm with reweighted $\mathscr{L}_1$ norm to make problem convex (some elements of $\boldsymbol{\alpha}$ will become 0)

*A. Tsolovikos et al. ``Estimation and Control of Fluid Flows Using Sparsity-Promoting Dynamic Mode Decomposition''. In: *IEEE Control Systems Letters* 5.4 (2021), pp. 1145–1150.

Sparse Reduced-Order Dynamics

- For some weighting factor ϵ:

$$\boldsymbol{\alpha} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \longrightarrow \begin{array}{c} \tilde{\Phi} = \begin{bmatrix} \phi_1 & \phi_2 & \cancelto{0}{\phi_3} & \cdots & \cancelto{0}{\phi_{q-1}} & \phi_q \end{bmatrix} \\ \tilde{\Lambda} = \mathrm{diag}\{\begin{bmatrix} \lambda_1 & \lambda_2 & \cancelto{0}{\lambda_3} & \cdots & \cancelto{0}{\lambda_{q-1}} & \lambda_q \end{bmatrix}\} \\ \mathbf{y}_k \approx \phi_1\psi_{1,k} + \phi_2\psi_{2,k} + \cancelto{0}{\phi_3}\psi_{3,k} + \cdots + \cancelto{0}{\phi_{q-1}}\psi_{q-1,k} + \phi_q\psi_{q,k} \end{array}$$

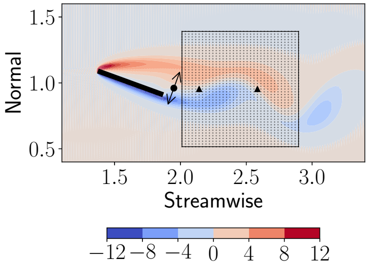
- In general, $n_x \leq q$ DMD modes will survive
- In complex modal form:

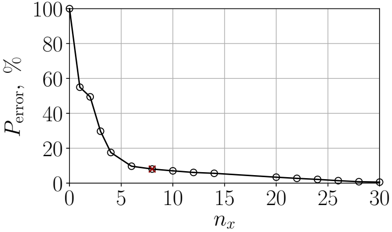
$$\begin{aligned} \tilde{\psi}_{k+1} &= \tilde{\Lambda}\tilde{\psi}_k + \tilde{\Gamma}\mathbf{u}_k \\ y_k &\approx \tilde{\Phi}\tilde{\psi}_k \end{aligned} \longrightarrow$$

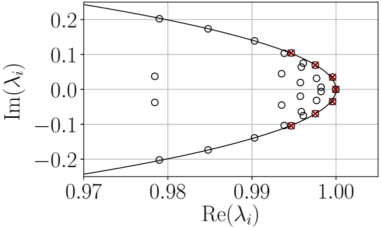
- In real modal form:

$$\boxed{\begin{aligned} \mathbf{x}_{k+1} &= A\mathbf{x}_k + B\mathbf{u}_k \\ \mathbf{y}_k &\approx \Theta\mathbf{x}_k \end{aligned}}$$

Application: Flat Plate Wake Stabilization





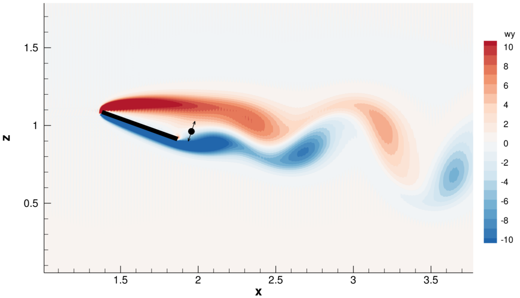


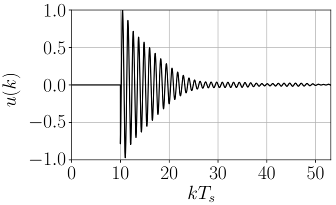
- $Re = 250$
- $n_y = 35 \times 43 = 1505$ grid points
- $p = 1501$ snapshots are used
- Start with $q = 30$ POD modes (99.5% energy)
- Keep only $n_x = 8$ DMD modes

Application: Flat Plate Wake Stabilization

- Online measurements $\mathbf{z}_k$: $n_z = 2$ out of $n_y = 1505$
- Estimate reduced-order state $\mathbf{x}_k$ from $\mathbf{z}_k$ using a Kalman Filter $\longrightarrow \hat{\mathbf{x}}_k$
- Stabilize flow using infinite-horizon LQG:
  - Stabilize only the 2 dominant modes (at the shedding frequency)
  - Policy: $\mathbf{u}_k^* = K\hat{\mathbf{x}}_k$ than minimizes

$$\mathscr{J}(\pi) = \lim_{N\to+\infty} \frac{1}{N}\mathbb{E}\left[\sum_{k=0}^{N-1} \mathbf{x}_k^\top Q_r \mathbf{x}_k + \mathbf{u}_k^\top R_r \mathbf{u}_k\right],$$





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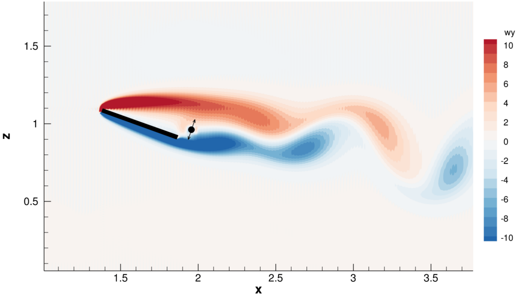

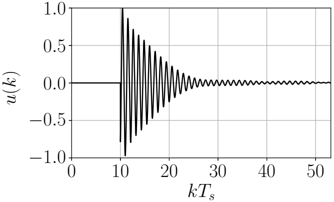

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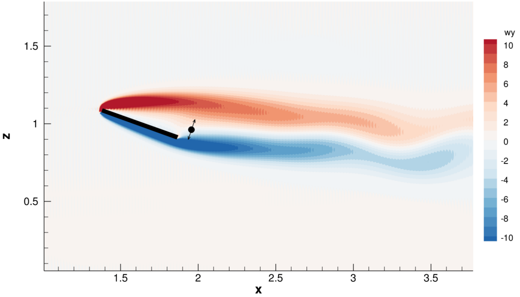

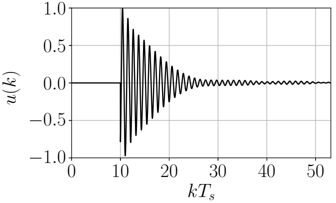

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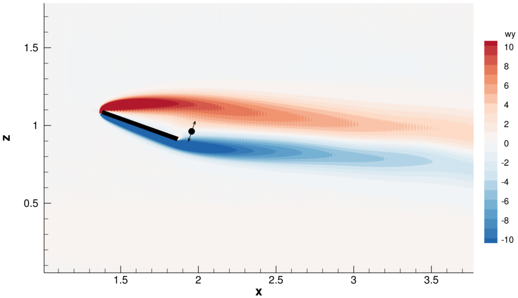

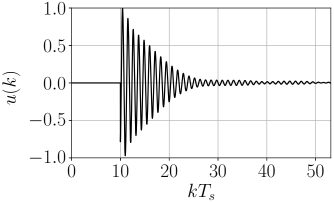

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