Multiple Model Dynamic Mode Decomposition for Flowfield and Parameter Estimation

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Problem Formulation

- Consider the state of a high-dimensional system of the form

$$\mathbf{y}_{k+1} = f(\mathbf{y}_k, m)$$

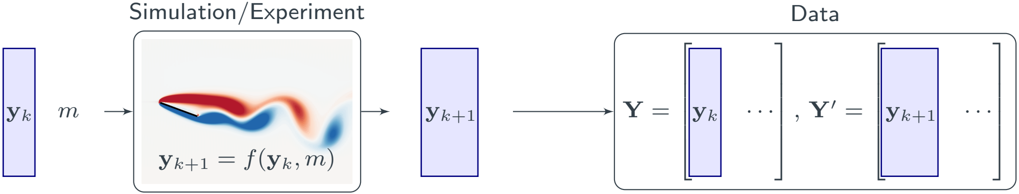
  where:
  - $\mathbf{y}_k \in \mathbb{R}^{n_y}$: high-dimensional state (e.g. flowfield)
  - $m \in \mathcal{M} \subseteq \mathbb{R}$: model parameter (e.g. angle of attack, pressure gradient, etc.)
- Goal: estimate in real time the full state $\mathbf{y}_k$ when
  - We can only measure $\mathbf{z}_k \in \mathbb{R}^{n_z}$ – a small subset of the elements of $\mathbf{y}_k$
  - The model parameter m is unknown
- Model Reduction using Dynamic Mode Decomposition:
  - Find a low-dimensional linear state space model using only data

Proposed approach:

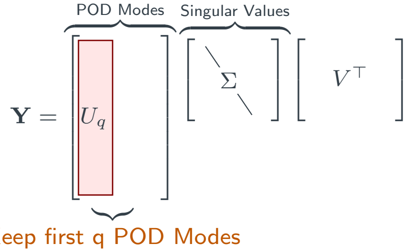
Split the parameter space $\mathcal{M}$ into subsets, train a DMD model on each subset (e.g. a model for different AoA) and perform multiple model estimation

Model Reduction using Dynamic Mode Decomposition

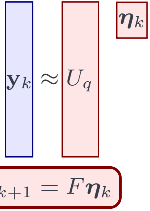
- Collect data for different model parameters m:



- Proper Orthogonal Decomposition



- Low-dimensional linear subspace spanned by the columns of U_q:



Model Reduction using Dynamic Mode Decomposition

- Fit linear dynamics for $\boldsymbol{\eta}_k$:

$$
\begin{aligned}
\boldsymbol{\eta}_{k+1} &= F\boldsymbol{\eta}_k \Rightarrow \\
U_q^\top \mathbf{y}_{k+1} &= F U_q^\top \mathbf{y}_k \Rightarrow \\
U_q^\top \mathbf{Y}' &= F U_q^\top \mathbf{Y} \Rightarrow \\
F &= U_q^\top \mathbf{Y}' \left(U_q^\top \mathbf{Y}\right)^\dagger
\end{aligned}
$$

- Eigenvalue decomposition of F:

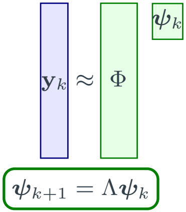
$$FW = W\Lambda$$

  - Eigenvectors: $W = \begin{bmatrix} \mathbf{w}_1 & \cdots & \mathbf{w}_q \end{bmatrix}$
  - Eigenvalues: $\Lambda = \mathrm{diag}\{\left[\lambda_1 \cdots \lambda_q\right]\}$

- DMD Modes:

$$\Phi = U_q W$$

- DMD Dynamics:



where

$$\boldsymbol{\psi}_k = W^{-1}\boldsymbol{\eta}_k \approx W^{-1} U_q y_k$$

Sparsity-Promoting DMD*

- Since Λ is diagonal, we can write:

$$\mathbf{y}_{k+1} \approx \Phi\Lambda\boldsymbol{\psi}_k = \sum_{i=1}^{q} \boldsymbol{\phi}_i \lambda_i \boldsymbol{\psi}_{i,k}$$

- Weight the contribution of each DMD mode $\boldsymbol{\phi}_i$ by $\alpha_i = 1$:

$$\mathbf{y}_{k+1} \approx \sum_{i=1}^{q} \alpha_i \boldsymbol{\phi}_i \lambda_i \boldsymbol{\psi}_{i,k}$$

- Stack everything together:

$$\mathbf{Y}' \approx \Phi \mathrm{diag}\{\boldsymbol{\alpha}\}\mathbf{R}, \quad \mathbf{R} = \Lambda\Phi^{\dagger}\mathbf{Y}$$

- Sparsity-Promoting Optimization:

$$\min_{\boldsymbol{\alpha}} \|\mathbf{Y}' - \Phi \mathrm{diag}\{\boldsymbol{\alpha}\}\mathbf{R}\|_{\mathrm{F}}^2 + \varepsilon \|\boldsymbol{\alpha}\|_0$$

  - Approximate data with linear dynamics
  - Promote sparsity: approximate $\mathscr{L}_0$ norm with reweighted $\mathscr{L}_1$ norm to make problem convex (some elements of $\boldsymbol{\alpha}$ will become 0)

*Tsolovikos et al., "Estimation and Control of Fluid Flows Using Sparsity-Promoting Dynamic Mode Decomposition".

Sparse Reduced-Order Dynamics

- For some weighting factor ϵ:

$$\boldsymbol{\alpha} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \longrightarrow \begin{array}{c} \tilde{\Phi} = \begin{bmatrix} \boldsymbol{\phi}_1 & \boldsymbol{\phi}_2 & \cancelto{0}{\boldsymbol{\phi}_3} & \cdots & \cancelto{0}{\boldsymbol{\phi}_{q-1}} & \boldsymbol{\phi}_q \end{bmatrix} \\ \tilde{\Lambda} = \mathrm{diag}\{\begin{bmatrix} \lambda_1 & \lambda_2 & \cancelto{0}{\lambda_3} & \cdots & \cancelto{0}{\lambda_{q-1}} & \lambda_q \end{bmatrix}\} \\ \mathbf{y}_k \approx \boldsymbol{\phi}_1 \psi_{1,k} + \boldsymbol{\phi}_2 \psi_{2,k} + \cancelto{0}{\boldsymbol{\phi}_3} \psi_{3,k} + \cdots + \cancelto{0}{\boldsymbol{\phi}_{q-1}} \psi_{q-1,k} + \boldsymbol{\phi}_q \psi_{q,k} \end{array}$$

- In general, $n_x \leq q$ DMD modes will survive
- In complex modal form:

$$\tilde{\boldsymbol{\psi}}_{k+1} = \tilde{\Lambda} \tilde{\boldsymbol{\psi}}_k$$
$$y_k \approx \tilde{\Phi} \tilde{\psi}_k$$

$$\longrightarrow$$

- In real modal form:

$$\boxed{\begin{aligned} \mathbf{x}_{k+1} &= A \mathbf{x}_k \\ \mathbf{y}_k &\approx \Theta \mathbf{x}_k \end{aligned}}$$

Flowfield Estimation using a Single DMD Model

- Estimate the entire flowfield $\mathbf{y}_k$ from limited measurements $\mathbf{z}_k$

$$\mathbf{z}_k = E_z \mathbf{y}_k$$

where $\mathbf{z}_k$ is a small subset of the elements of $\mathbf{y}_k$.

- Reduced-order dynamics:

$$\begin{aligned} \mathbf{x}_{k+1} &= A\mathbf{x}_k + \mathbf{w}_k, \\ \mathbf{z}_k &= C\mathbf{x}_k + \mathbf{v}_k, \end{aligned}$$

where $\mathbf{w}_k \sim \mathcal{N}(0, Q)$ is the process noise and $\mathbf{v}_k \sim \mathcal{N}(0, R)$ is the measurement noise.

Measure $\mathbf{z}_k \rightarrow$ **Estimate** $\mathbf{x}_k \rightarrow$ **Reconstruct** $\mathbf{y}_k$

Flowfield Estimation using a Single DMD Model

- Kalman Filter
  - State Prediction:

$$
\begin{aligned}
\hat{\mathbf{x}}_k^- &= A\hat{\mathbf{x}}_{k-1} \\
P_k^- &= AP_{k-1}A^\top + Q_e
\end{aligned}
$$

  - Measurement Update:

$$
\begin{aligned}
K_k &= P_k^- C^\top \left[CP_k^- C^\top + R_e\right]^{-1} \\
\hat{\mathbf{x}}_k &= \hat{\mathbf{x}}_k^- + K_k \left[\mathbf{z}_k - C\hat{\mathbf{x}}_k^-\right] \\
P_k &= \left[I - K_k C\right] P_k^-
\end{aligned}
$$

Flowfield:

$$
\hat{\mathbf{y}}_k = \Theta\hat{\mathbf{x}}_k
$$

- Estimate error covariances from data:

$$
\begin{aligned}
\mathbf{w}_k &= \Theta^\dagger \mathbf{y}_{k+1} - A\Theta^\dagger \mathbf{y}_k \rightarrow Q = \mathbb{E}\left[\mathbf{w}_k \mathbf{w}_k^\top\right] \\
\mathbf{v}_k &= \mathbf{z}_k - C\Theta^\dagger \mathbf{y}_k \rightarrow R = \mathbb{E}\left[\mathbf{v}_k \mathbf{v}_k^\top\right] + \sigma_z I
\end{aligned}
$$

Flowfield Estimation using a Multiple DMD Model

- If dynamics depend on an (unknown) model parameter $m \in \mathcal{M} \subseteq \mathbb{R}$:

$$\mathbf{y}_{k+1} = \mathbf{f}(\mathbf{y}_k; m)$$

- Assume that we also have M sets of measurements:

$$\mathcal{D}_i = \{(\mathbf{y}^{(j)}, \mathbf{f}(\mathbf{y}^{(j)}; m_i) : \ j = 1, \ldots, p\},$$

- Compute a sparse DMD model for each dataset:

$$\begin{aligned} \mathbf{x}_{k+1}^{(i)} &= A^{(i)}\mathbf{x}_k^{(i)} + \mathbf{w}_k^{(i)}, \quad & \mathbf{w}_k^{(i)} &\sim \mathcal{N}(0, Q^{(i)}) \\ \mathbf{y}_k &= \Theta^{(i)}\mathbf{x}_k^{(i)} + \boldsymbol{\epsilon}_k^{(i)}, \quad & \boldsymbol{\epsilon}_k^{(i)} &\sim \mathcal{N}(0, \Sigma^{(i)}) \\ \mathbf{z}_k &= C^{(i)}\mathbf{x}_k^{(i)} + \mathbf{v}_k^{(i)}, \quad & \mathbf{v}_k^{(i)} &\sim \mathcal{N}(0, R^{(i)}) \end{aligned}$$

M DMDsp models for M different parameters m_i

Flowfield Estimation using a Multiple DMD Model

- Minimum mean-squared error estimate:

$$\hat{\mathbf{y}}_k^{\text{MMSE}} = \sum_{i=1}^{M} \mu_k^{(i)} \Theta^{(i)} \hat{\mathbf{x}}_k^{(i)},$$

- Model weights: probability of ith model being correct

$$\mu_k^{(i)} = p(m = m_i \mid Z^k)$$

- Multiple Model Kalman Filter:
  - Run M Kalman filters – one for each model
  - Update model weights:

$$\mu_k^{(i)} = p(m = m_i \mid Z^k) = \frac{p(\mathbf{z}_k \mid m = m_i, Z^{k-1}) \mu_{k-1}^{(i)}}{\sum_{j=1}^{M} p(\mathbf{z}_k \mid m = m_j, Z^{k-1}) \mu_{k-1}^{(j)}}$$

where

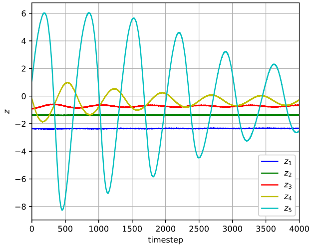
$$p(\mathbf{z}_k \mid m = m_i, Z^{k-1}) = \mathcal{N}(\nu_k^{(i)}; 0, S_k^{(i)})$$

Example: Blasius Boundary Layer with Varying APG

- 2D Blasius boundary layer at $Re_\delta = 1400$
- Parameter m: Adverse pressure gradient $dp/dx < 0$
- Training datasets for:
  - $m_1 = -12 \times 10^{-3}$
  - $m_2 = -16 \times 10^{-3}$
  - $m_3 = -20 \times 10^{-3}$
  - $m_4 = -24 \times 10^{-3}$
- High-dimensional state: vorticity at an orthogonal grid of size $141 \times 39 = 5499$
- $M = 4$ DMDsp models

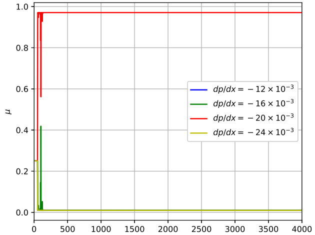
- Test dataset:
  - Measurements $\mathbf{z}_k$: vorticity at 5 locations near the wall
  - Parameter m is unknown

Measurements

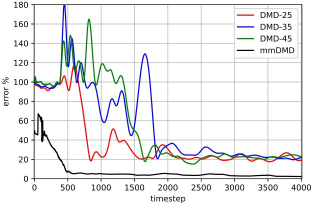


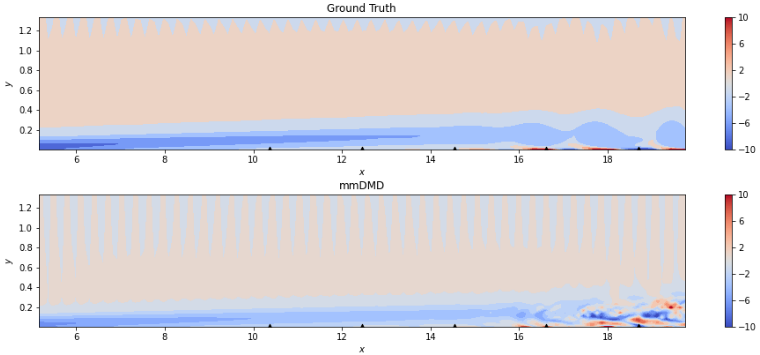
Example: Blasius Boundary Layer with Varying APG

Model Probabilities



Estimation Error



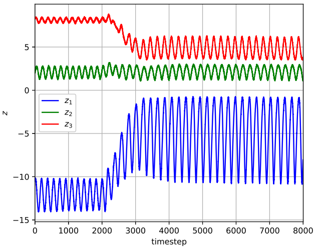


Example: Flat Plate with Varying AoA

- Inclined flat plate at $Re = 250$
- Parameter m: Angle of attack (AoA)
- Training datasets for:
  - $m_1 = 20^o$ (constant AoA)
  - $m_2 \in (20^o, 30^o)$ (varying AoA)
  - $m_3 = 30^o$ (constant AoA)
- High-dimensional state: vorticity at an orthogonal grid of size $231 \times 135 = 31185$
- $M = 3$ DMDsp models

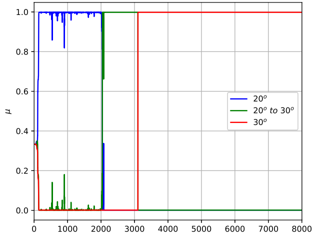
- Test dataset:
  - Measurements $\mathbf{z}_k$: vorticity at 3 locations in the wake
  - Parameter m is unknown and slowly varying

Measurements

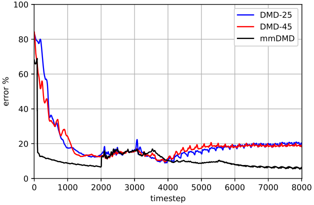


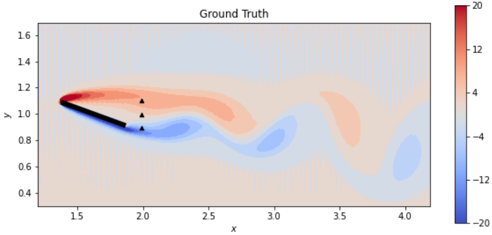
Example: Flat Plate with Varying AoA

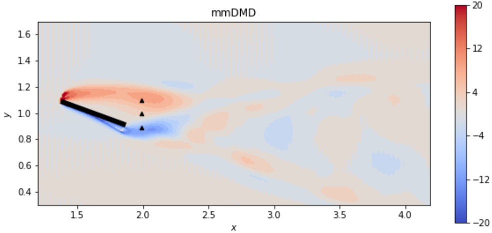
Model Probabilities



Estimation Error







Conclusions and Future Work

- We presented a framework for flowfield estimation using multiple DMD models in settings with unknown parameters
- The multiple model approach chooses the DMD model matching the measurements best
- Unknown parameters can also be inferred
- Next steps: use multiple-model approach for closed-loop flow control

`alextsolovikos.github.io`

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